

1. Basic Equations

1.1. Heat Flux - Fourier Law

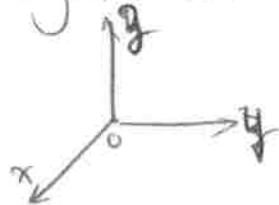
For homogeneous, isotropic solid:

$$\boxed{\vec{q}(\vec{r}, t) = -k \nabla T(\vec{r}, t)}$$

↑ ↑
heat flux thermal conductivity

\vec{q} heatflow per unit time, per unit area of the isothermal surfaces
Vector — points in the direction of decreasing temperature

* rectangular coordinates:



$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

$$\begin{cases} q_x = -k \frac{\partial T}{\partial x} \\ q_y = -k \frac{\partial T}{\partial y} \\ q_z = -k \frac{\partial T}{\partial z} \end{cases}$$

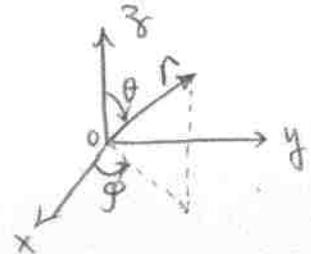
* cylindrical coordinates:



$$\begin{cases} x = r \cos \phi \\ y = r \sin \phi \\ z = z \end{cases}$$

$$\begin{cases} q_r = -k \frac{\partial T}{\partial r} \\ q_\theta = -k \frac{1}{r} \frac{\partial T}{\partial \theta} \\ q_z = -k \frac{\partial T}{\partial z} \end{cases}$$

* spherical coordinates:



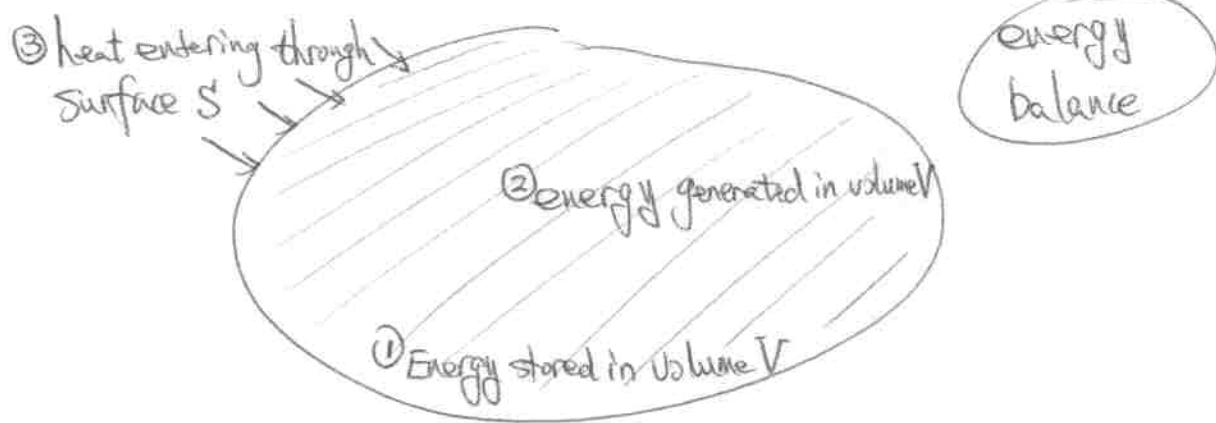
$$\begin{cases} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{cases}$$

$$\begin{cases} q_r = -k \frac{\partial T}{\partial r} \\ q_\theta = -k \frac{1}{r \sin \theta} \frac{\partial T}{\partial \theta} \\ q_\phi = -k \frac{1}{r} \frac{\partial T}{\partial \phi} \end{cases}$$

1.2. Heat Conduction Equation — isotropic solid

For homogeneous, isotropic solid, with heat generation
 electrical, optical, chemical, ...

* Consider a control volume V with surface area S .



$$\left\{ \begin{array}{l} \text{rate of change of stored energy in } V = \int_V \rho c_p \frac{\partial T(\vec{r}, t)}{\partial t} dV \\ \text{rate of energy generated in } V = \int_V g(\vec{r}, t) dV \\ \text{rate of heat entering through } S = - \oint_S \vec{q}(\vec{r}, t) \cdot d\vec{S} \\ \quad = - \int_V \nabla \cdot \vec{q}(\vec{r}, t) dV \end{array} \right.$$

Energy Balance:

$$-\int_V \nabla \cdot \vec{q}(\vec{r}, t) dV + \int_V g(\vec{r}, t) dV = \int_V \rho c_p \frac{\partial T(\vec{r}, t)}{\partial t} dV$$

$$\boxed{-\nabla \cdot \vec{q}(\vec{r}, t) + g(\vec{r}, t) = \rho c_p \frac{\partial T(\vec{r}, t)}{\partial t}}$$

Substituting: $\vec{g} = -k \nabla T$

$$\nabla \cdot [k \nabla T(\vec{r}, t)] + g(\vec{r}, t) = \rho q \frac{\partial T(\vec{r}, t)}{\partial t}$$

For $k = \text{Const.}$ — independent of \vec{r} and t :

$$k \nabla^2 T(\vec{r}, t) + g(\vec{r}, t) = \rho q \frac{\partial T(\vec{r}, t)}{\partial t}$$

define diffusivity $\alpha = \frac{k}{\rho c_p}$

$$\boxed{\nabla^2 T(\vec{r}, t) + \frac{1}{\alpha} g(\vec{r}, t) = \frac{1}{\alpha} \frac{\partial T(\vec{r}, t)}{\partial t}}$$

without heat generation: $g = 0$

$$\boxed{\nabla^2 T(\vec{r}, t) = \frac{1}{\alpha} \frac{\partial T(\vec{r}, t)}{\partial t}}$$

for steady-state and without heat generation:

$$\boxed{\nabla^2 T(\vec{r}) = 0}$$

Rectangular coordinates: $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{1}{k} g = \frac{1}{\alpha} \frac{\partial T}{\partial t}$

Cylindrical coordinates: $\frac{1}{r^2} \frac{\partial^2 (r^2 T)}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 T}{\partial z^2} + \frac{1}{k} g = \frac{1}{\alpha} \frac{\partial T}{\partial t}$

Spherical coordinates: $\frac{1}{r^2} \frac{\partial^2 (r^2 T)}{\partial r^2} + \frac{1}{r^2 \sin \theta} \frac{\partial^2 (r^2 \sin \theta T)}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2} + \frac{1}{k} g = \frac{1}{\alpha} \frac{\partial T}{\partial t}$

1.3. Heat Conduction Equation — Anisotropic Solid

* thermal conductivity — 2nd order tensor:

$$\underline{K} = \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix} \quad (k_{ij} = k_{ji}) \quad \begin{matrix} \text{Symmetric!} \\ \text{It can be proved by} \\ \underline{\text{Onsager's principle}} \end{matrix}$$

* heat flux:

$$\begin{pmatrix} q \\ \frac{\partial q_x}{\partial x} \\ \frac{\partial q_y}{\partial y} \\ \frac{\partial q_z}{\partial z} \end{pmatrix} = - \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix} \begin{pmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \\ \frac{\partial T}{\partial z} \end{pmatrix}$$

$$\left\{ \begin{array}{l} q_x = -K_{11} \frac{\partial T}{\partial x} - K_{12} \frac{\partial T}{\partial y} - K_{13} \frac{\partial T}{\partial z} \\ q_y = -K_{21} \frac{\partial T}{\partial x} - K_{22} \frac{\partial T}{\partial y} - K_{23} \frac{\partial T}{\partial z} \\ q_z = -K_{31} \frac{\partial T}{\partial x} - K_{32} \frac{\partial T}{\partial y} - K_{33} \frac{\partial T}{\partial z} \end{array} \right.$$

* heat conduction equation:

$$-\nabla \cdot \vec{q} + q = \rho c_p \frac{\partial T}{\partial t}$$

$$-\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right) + q = \rho c_p \frac{\partial T}{\partial t}$$

$$K_{11} \frac{\partial^2 T}{\partial x^2} + K_{12} \frac{\partial^2 T}{\partial y^2} + K_{13} \frac{\partial^2 T}{\partial z^2} + (K_{12} + K_{21}) \frac{\partial^2 T}{\partial xy} + (K_{13} + K_{31}) \frac{\partial^2 T}{\partial xz} + (K_{23} + K_{32}) \frac{\partial^2 T}{\partial yz}$$

$$+ q(x, y, z, t) = \rho c_p \frac{\partial T(x, y, z, t)}{\partial t}$$

* Determination of Principal Axes

$$(x, y, z) \rightarrow (\xi, \eta, \zeta)$$

Coordinate transformation

	$0X$	$0Y$	$0Z$
0ξ	l_1	m_1	n_1
0η	l_2	m_2	n_2
0ζ	l_3	m_3	n_3

(directional cosine)

In New coordinate system (ξ, η, ζ):

$$K = \begin{pmatrix} k_\xi & 0 & 0 \\ 0 & k_\eta & 0 \\ 0 & 0 & k_\zeta \end{pmatrix}$$

$$k_\xi \frac{\partial^2 T}{\partial \xi^2} + k_\eta \frac{\partial^2 T}{\partial \eta^2} + k_\zeta \frac{\partial^2 T}{\partial \zeta^2} + g = \rho C_p \frac{\partial T}{\partial t}$$

Find (k_ξ, k_η, k_ζ):

k_ξ, k_η, k_ζ are eigenvalues of equation
(three roots)

$$\begin{vmatrix} K_{11}-\lambda & K_{12} & K_{13} \\ K_{21} & K_{22}-\lambda & K_{23} \\ K_{31} & K_{32} & K_{33}-\lambda \end{vmatrix} = 0$$

Find principal axes (directional cosines):

$$\left\{ \begin{array}{l} \text{for } \lambda_1 = k_\xi, \text{ find } \begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix} \text{ from: } \begin{cases} \begin{pmatrix} K_{11}-k_\xi & K_{12} & K_{13} \\ K_{21} & K_{22}-k_\xi & K_{23} \\ K_{31} & K_{32} & K_{33}-k_\xi \end{pmatrix} \begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix} = 0 \\ l_1^2 + m_1^2 + n_1^2 = 1 \end{cases} \\ \text{for } \lambda_2 = k_\eta, \text{ find } \begin{pmatrix} l_2 \\ m_2 \\ n_2 \end{pmatrix} \text{ from: } \begin{cases} \begin{pmatrix} K_{11}-k_\eta & K_{12} & K_{13} \\ K_{21} & K_{22}-k_\eta & K_{23} \\ K_{31} & K_{32} & K_{33}-k_\eta \end{pmatrix} \begin{pmatrix} l_2 \\ m_2 \\ n_2 \end{pmatrix} = 0 \\ l_2^2 + m_2^2 + n_2^2 = 1 \end{cases} \\ \text{for } \lambda_3 = k_\zeta, \text{ find } \begin{pmatrix} l_3 \\ m_3 \\ n_3 \end{pmatrix} \text{ from: } \begin{cases} \begin{pmatrix} K_{11}-k_\zeta & K_{12} & K_{13} \\ K_{21} & K_{22}-k_\zeta & K_{23} \\ K_{31} & K_{32} & K_{33}-k_\zeta \end{pmatrix} \begin{pmatrix} l_3 \\ m_3 \\ n_3 \end{pmatrix} = 0 \\ l_3^2 + m_3^2 + n_3^2 = 1 \end{cases} \end{array} \right.$$

1.4. Lumped System Formulation — Simplification

General conduction equation: $\nabla^2 T(\vec{r}, t) + \frac{1}{k} q(\vec{r}, t) = \frac{1}{\rho c} \frac{\partial T(\vec{r}, t)}{\partial t}$

temperature (T) is a function of \vec{r} .

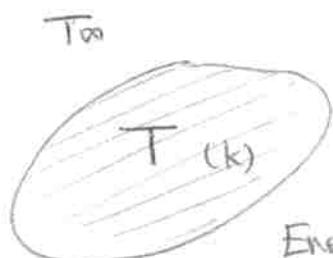
* Special case: (Without heat generation)

Temperature variation of solid can be neglected!

When: internal thermal resistance (conduction) is smaller than external thermal resistance (convection).

$$\text{Biot Number: } Bi = \frac{\text{(internal thermal resistance)}}{\text{(external thermal resistance)}} \text{ (small)}$$

$$= \frac{\left(\frac{L}{KA}\right)}{\left(\frac{1}{hA}\right)} \quad \begin{matrix} \leftarrow \text{Conduction} \\ \leftarrow \text{Convection} \end{matrix}$$



Energy Balance:

$$\begin{cases} \text{Volume } V \\ \text{Area } A \\ \text{length scale } L \sim \frac{V}{A} \end{cases}$$

$$\begin{cases} \text{rate of change of stored energy} = \rho C_p V \frac{dT(t)}{dt} \\ \text{rate of change of energy loss} = hA [T(t) - T_{\infty}] \end{cases}$$

$$\rho C_p V \frac{dT(t)}{dt} = -hA [T(t) - T_{\infty}]$$

i.e.

$$\begin{cases} \frac{dT(t)}{dt} + \frac{hA}{\rho C_p V} [T(t) - T_{\infty}] = 0 & \text{(for } t > 0\text{)} \\ T(0) = T_0 & \text{(for } t = 0\text{)} \end{cases}$$

Solution: $T(t) = T_{\infty} + (T_0 - T_{\infty}) e^{-\frac{hA}{\rho C_p V} t}$

(Valid for $Bi < 0.1$)